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Approximation by weighted polynomials

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Abstract

It is proven that if $xQ'(x)$ is increasing on $(0, +\infty)$ and $w(x) = \exp(-Q(x))$ is the corresponding weight on $[0, +\infty)$, then every continuous function that vanishes outside the support of the extremal measure associated with w can be uniformly approximated by weighted polynomials of the form $w^n P_n$. This problem was raised by Totik, who proved a similar result (the Borwein–Saff conjecture) for convex Q. A general criterion is introduced, too, which guarantees that the support of the extremal measure is an interval. With this criterion we generalize the above approximation theorem as well as that one, where Q is supposed to be convex.

1. Introduction

In [6], Totik settled a basic conjecture in the theory of approximation by weighted polynomials of the form $w^n P_n$, where $w = \exp(-Q)$ is a given weight function. This conjecture was the Borwein–Saff conjecture and it was stated for convex Q.

In this paper, the same theorem is stated and proved, but for more general Q. The criterion we use covers the case when (a) Q is convex, and also the case when (b) Q is defined on $[0, +\infty)$ and $xQ'(x)$ is increasing. In logarithmic potential theory many theorems use either (a) or (b) as their assumption on Q, since both of them guarantee that the support S_w is an interval. The criterion to be introduced is weaker than both (a) and (b), but still guarantees that the support S_w is an interval. (Furthermore, we merely assume that Q is absolutely continuous and not necessarily differentiable.) Thus, this criterion itself is a useful result of this paper.

The reader can find the definition of the logarithmic capacity in [4, I.1]. We say that a property holds quasi-everywhere, if the set where it does not hold has capacity 0.

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Let $\Sigma \subset \mathbb{R}$ be a closed set. (We assume that Σ is regular with respect to the Dirichlet problem in $\mathbf{C} \setminus \Sigma$.) A weight function w on Σ is said to be admissible, if it satisfies the following three conditions:

(i) w is upper semi-continuous,
(ii) $\{x \in \Sigma\colon\ w(x) > 0\}$ has positive capacity,
(iii) if Σ is unbounded, then $|z|w(z) \to 0$ as $|z| \to \infty,\ z \in \Sigma$.

Unless otherwise noted, we will always assume in the theorems that w is continuous.

We define Q by

$$w(x) =: \exp(-Q(x)),$$

so $Q : \Sigma \to (-\infty, \infty]$ is a lower semi-continuous function. This Q is called the external field.

Let $\mathcal{M}(\Sigma)$ be the collection of all positive unit Borel measures with compact support in Σ. We define the *logarithmic potential* of $\mu \in \mathcal{M}(\Sigma)$ as

$$U^{\mu}(x) := \int \log \frac{1}{|x-t|}\, d\mu(t),$$

and the *weighted energy integral* as

$$I_w(\mu) := -\int\int \log(|x-y|w(x)w(y))\, d\mu(x)\, d\mu(y).$$

We will need the following basic theorem [4, Theorem I.1.3]:

Theorem A. *Let w be an admissible* (*not necessarily continuous*) *weight on the closed set Σ and let $V_w := \inf\{I_w(\mu) : \mu \in \mathcal{M}(\Sigma)\}$. Then*

(a) *V_w is finite,*
(b) *there exists a unique element $\mu_w \in \mathcal{M}(\Sigma)$ such that $I_w(\mu_w) = V_w$,*
(c) *setting $F_w := V_w - \int Q\, d\mu_w$, the inequality*

$$U^{\mu_w}(x) + Q(x) \geqslant F_w$$

holds quasi-everywhere on Σ,

(d) *the inequality*

$$U^{\mu_w}(x) + Q(x) \leqslant F_w \tag{1}$$

holds for all $x \in S_w := \mathrm{supp}(\mu_w)$.

Remark. According to our definition, every measure in $\mathcal{M}(\Sigma)$ has compact support, so the support S_w is a compact set.

The measure μ_w is called *the equilibrium* or *extremal measure* associated with w.

Notation 1. *When we say that a property holds* inside G*—where G is a subset of $\mathbb{R}$—we mean that the property is satisfied on every compact subset of G.*

We are considering uniform approximation of continuous functions on Σ by weighted polynomials of the form $w^n P_n$, where $deg\, P_n \leqslant n$. Theorem B is a Stone–Weierstrass-type theorem for this kind of approximation (see [1] or [4, Theorem VI.1.1]). Unless otherwise noted, we will always assume in the theorems that w is continuous.

Theorem B. *There exists a closed set $Z = Z(w) \subset \Sigma$, such that a continuous function f on Σ is the uniform limit of weighted polynomials $w^n P_n$, $n = 1, 2, \ldots$, if and only if f vanishes on Z.*

Thus, the problem of what functions can be approximated is equivalent to determining what points lie in $Z(w)$. This latter problem is intimately related to the density of μ_w. The support $S_w := supp(\mu_w)$ plays a special role (see [5], or [1] or see [4, Theorem VI.1.2]):

Theorem C. $Z \supset \Sigma \setminus (int\, S_w)$.

The Borwein–Saff conjecture was that $Z = \Sigma \setminus (int\, S_w)$, if Q is convex, i.e., a continuous function f can be approximated by weighted polynomials $w^n P_n$, if and only if f vanishes outside S_w. This was proved by Totik [6], see Theorem G. The idea of the proof was based on the following definition and theorems:

Definition D. We say that a non-negative function v has smooth integral on an interval $[a, b]$, if for every $0 < \varepsilon$ there is a $0 < \delta$, such that if $I, J \subset [a, b]$ are two adjacent intervals of equal length smaller than δ, then

$$\int_I v \leqslant (1 + \varepsilon) \int_J v.$$

Clearly, all positive continuous functions have smooth integral, but also $\log(1/|t|)$ has smooth integral, on any interval $[-a, a]$, where $0 < a < 1$. (We will prove it later.)

Theorem E. *Let us suppose that* $[a,b]$ *is a subset of the support* S_w, *and the extremal measure has a density* v *on* $[a,b]$ *that has a positive lower bound and smooth integral there. Then* $(a,b)\cap Z(w)=\emptyset$.

Theorem F. *Let us suppose that* (a,b) *is a subset of the support* S_w, *and that* Q *is convex on* (a,b). *Then* μ_w *has a density* v *in* (a,b) *which has a positive lower bound and smooth integral inside* (a,b).

From these two theorems follows:

Theorem G. *Let us suppose that* (a,b) *is a subset of the support* S_w, *and that* Q *is convex on* (a,b). *Then* $(a,b)\cap Z(w)=\emptyset$. *In particular, every function that vanishes outside* (a,b) *can be uniformly approximated by weighted polynomials of the form* w^nP_n.

Notice that Theorem G (as well as its generalization, Theorem 5) is a local result; it works for any part of the extremal support where Q is convex.

Totik raised the question whether Theorem G is still valid if $\Sigma=[0,+\infty)$ and if we replace the convexity condition with the condition that $xQ'(x)$ is increasing. Theorem 5 answers this question positively.

2. Main results

Definition 2. We say that a function $Q: D\to\bar{\mathbb{R}}$ $(D\subset\mathbb{R})$ is weak convex on an interval $I=[(a,b)]\subset D$ $(a,b\in\bar{\mathbb{R}})$ with basepoints $A,B\in\bar{\mathbb{R}}$, $A<B$, if the following properties hold:

(i) $I\subset[A,B]$,

(ii) Q is absolutely continuous inside (a,b) (so Q' exists a.e. in I),

(iii) if $a\in I$, then

$$\lim_{x\to a+0} Q(x)\leq Q(a),$$

and if $b\in I$, then

$$\lim_{x\to b-0} Q(x)\leqslant Q(b),$$

(iv) for some $\kappa\in[a,b]$:

$$Q'(x)\leqslant 0 \quad \text{on} \quad (a,\kappa),$$

$$Q'(x)\geqslant 0 \quad \text{on} \quad (\kappa,b).$$

(v) $(B-x)Q'(x)$ is increasing on (a,κ),

(vi) $(x-A)Q'(x)$ is increasing on (κ, b).

If $B=+\infty$, we replace (v) by

(v)* $Q'(x)$ is increasing on (a, κ), and if $A=-\infty$, we replace (vi) by
(vi)* $Q'(x)$ is increasing on (κ, b).
We will simply just say that Q is weak convex on an interval I (without mentioning the basepoints), if Q is weak convex on the interval $I=[(a,b)]\subset D$ $(a,b\in\bar{\mathbb{R}})$ with basepoints a and b.

Remark. Since $\kappa=a$ (or $\kappa=b$) is allowed, it is also possible that Q' is everywhere positive (or negative) on I.

Notation 3. *Throughout the article (in the above definition as well as in the theorems) we agree on the following. Suppose that a function $f(x)$—which is usually defined by using $Q'(x)$—is said to be increasing (or decreasing) on a set E, but it is defined on a set $F\subset E$, where $E\setminus F$ has measure zero. Then by "$f(x)$ is increasing on E" we mean that there exists a set $G\subset F$ so that $E\setminus G$ has measure zero and $f(x)$ is increasing on G. In other words, we do not require that $f(x)$ is increasing everywhere, where it is defined. Similarly, if we write that, say, $Q'(x)>0$ on $(\kappa, B]$, we will mean that there exists a set $G\subset(\kappa, B]$ so that $(\kappa, B]\setminus G$ has measure 0, $Q'(x)$ exists on G and $Q'(x)>0$ on G. This agreement weakens the assumptions on the considered functions in the theorems, but the given proofs are correct for this modified increasing/decreasing definitions as well.*

Some immediate consequences of the above definition are:

Proposition 4. *Let $I\subset\mathbb{R}$ be an interval.*

(a) *If $I\subset[A_1,B_1]\subset[A_2,B_2]\subset\bar{\mathbb{R}}$ and Q is weak convex on I with basepoints A_2 and B_2, then Q is also weak convex on I with basepoints A_1 and B_1.*
(b) *A function Q is convex on I if and only if Q is weak convex on I with basepoints $-\infty$ and $+\infty$.*
(c) *Hence every convex function on I $(\subset[A_1,B_1])$ is also weak convex on I with basepoints A_1 and B_1. In particular, every convex function on I is also weak convex on I.*
(d) *If Q is absolutely continuous inside $(0,+\infty)$ and $xQ'(x)$ is increasing on $(0,+\infty)$, then Q is weak convex on $(0,+\infty)$.*

Thus "weak convexity" is a weaker condition than either the "Q is convex", or the "$xQ'(x)$ is increasing" conditions.

So the following theorem, which is our *main theorem*, is a generalization of Theorem G to a larger class of functions.

Theorem 5. *Let us suppose that (a,b) is a subset of the support S_w and that Q is weak convex on (a,b) with basepoints A, B satisfying $S_w \subset [A,B]$. Then $(a,b) \cap Z(w) = \emptyset$.*

In particular, every function that vanishes outside (a,b) can be uniformly approximated by weighted polynomials of the form $w^n P_n$.

Theorem 5 will follow from Theorem E, since we can generalize Theorem F as follows:

Theorem 6. *Let us suppose that (a,b) is a subset of the support S_w, and that Q is weak convex on (a,b) with basepoints A, B satisfying $S_w \subset [A,B]$. Then μ_w has a density v in (a,b) which has a positive lower bound and smooth integral inside (a,b).*

Because of Proposition 4, in Theorem 5 we should choose A and B as close to $\min S_w$ and $\max S_w$ (respectively) as possible to get the weakest assumption on Q. Thus if the values of $\min S_w$ and $\max S_w$ are known, the best choices are $A := \min S_w$ and $B := \max S_w$. If, however, we have no information at all of the locations of $\min S_w$ and $\max S_w$, we must choose $A := -\infty$ and $B := +\infty$ to be sure that $S_w \subset [A,B]$ is satisfied, and in this case Theorem 5 reduces to Theorem G: the assumption on Q is to be convex.

Notice, however, that $S_w \subset \Sigma$ is always true, so the following corollary is a special case of Theorem 5. The advantage of this corollary is that no information about $\min S_w$ and $\max S_w$ is needed to check whether the weak convexity assumption on a given Q is satisfied.

Corollary 7. *Let $(a,b) \subset S_w$. If Q is weak convex on (a,b) with basepoints* $\min \Sigma$ *and* $\max \Sigma$ *then $(a,b) \cap Z(w) = \emptyset$.*

In particular, when Σ is, say, lower bounded, and we write out the definition of weak convexity, we gain Corollary 8.

Corollary 8. *Let $\Sigma = [A, +\infty)$ be a half-line and let $(a,b) \subset S_w$. Suppose that Q is absolutely continuous inside (a,b). If for some $\kappa \in [a,b]$*

$$\begin{aligned} &Q(x) \text{ is decreasing and convex on } (a,\kappa), \\ &(x-A)Q'(x) \text{ is non-negative and increasing on } (\kappa, b), \end{aligned} \tag{2}$$

then $(a,b) \cap Z(w) = \emptyset$.

Remark. Corollary 8 implies that if $\Sigma = [0, +\infty)$, and

$$xQ'(x) \text{ is increasing on } (a,b) \subset S_w, \tag{3}$$

then $(a,b)\cap Z(w)=\emptyset$. Eq. (3) is an often used criterion in logarithmic potential theory (with the assumption that Q is differentiable). Notice that Corollary 8 assumes much less about Q, firstly Q does not have to be a differentiable just absolutely continuous, secondly on the interval (a,κ) (where $Q'(x)\leqslant 0$), (2) is a weaker assumption than (3).

Remark. One might ask whether we really need the $S_w\subset[A,B]$ assumption in Theorem 5. To see that we do, consider the following example. Let

$$v(t) := ac\chi_{[2,2.01]} + (1-a)c\frac{t^2}{\sqrt{1-t^2}} \quad t\in(-1,1)\cup[2,2.01] =: \Sigma,$$

where $\chi_{[2,2.01]}$ is the characteristic function of the interval $[2,2.01]$, $a\in(0,1)$, and $0<c$ is chosen so that $\int v=1$. If we set

$$Q(x) := \int \log|x-t|v(t)\,dt, \quad x\in\Sigma,$$

then by Theorem I.3.3 [4], the extremal measure μ_w associated with $w=\exp(-Q)$ has density $v(t)$. We can see easily, that if a is close to 1, then on $[-1/2,1/2]$ $0<Q'(x)$ and $(1+x)Q'(x)$ is increasing, however $v(t)\sim t^2$ as $t\to 0$, so by [2] (or [4, Theorem VI.1.8]), $0\in Z(w)$.

Now let us see some other theorems to demonstrate why the "weak convexity" is an appropriate generalization of both the "Q is convex", and the "$xQ'(x)$ is increasing" conditions. The following theorems are generalizations of some classical results (see [4, Sections IV.1 and IV.3]). We will also need to prove Theorem 5.

Theorem 9. *Let $w=\exp(-Q)$ be an admissible (not necessarily continuous) weight on $\mathbb{R}$ and suppose that Q is weak convex on the interval I with basepoints $A,B\in\bar{\mathbb{R}}$ satisfying $S_w\subset[A,B]$. Then $S_w\cap I$ is an interval.*

Remark. A weight defined on Σ and the same weight defined on $\mathbb{R}$ which is zero on $\mathbb{R}\setminus\Sigma$ gives us the same equilibrium measure. Hence, this theorem is as general as if $\mathbb{R}$ were replaced by Σ. (We avoided Σ in the formulation, because here we do not suppose that Σ is regular with respect to the Dirichlet problem in $\mathbf{C}\setminus\Sigma$.)

Theorem 10. *Let $w=\exp(-Q)$ be an admissible (not necessarily continuous) weight on $\mathbb{R}$ and suppose that Q is weak convex on the interval $[A,B]$ satisfying $S_w\subset[A,B]$. Then the support is a finite interval (by Theorem 9), say, $S_w=[a,b]$ and the endpoints a,b satisfy the following conditions:*

(i) *If* $b < B$, *then*

$$\frac{1}{\pi}\int_a^b \sqrt{\frac{x-a}{b-x}}\, Q'(x)\, dx = 1,$$

(ii) *if* $a > A$, *then*

$$\frac{1}{\pi}\int_a^b \sqrt{\frac{b-x}{x-a}}\, Q'(x)\, dx = -1,$$

(iii) *if* $b = B$, *then*

$$\frac{1}{\pi}\int_a^B \sqrt{\frac{x-a}{B-x}}\, Q'(x)\, dx \leqslant 1,$$

(iv) *if* $a = A$, *then*

$$\frac{1}{\pi}\int_A^b \sqrt{\frac{b-x}{x-A}}\, Q'(x)\, dx \geqslant -1.$$

In the next theorem, we give an integral representation for the density of the equilibrium measure μ_w. We emphasize that such integral representation was known only in the case when $xQ'(x)$ is increasing so far (see [4, Theorem IV.3.2]), not even in the case when Q is convex. Now Theorem 11 can also be applied to any convex Q which has bounded derivative, since every convex function is also weak convex.

In the proof we will define $v(t)$ almost everywhere, namely on a subset of $[-1,1]$ with full measure on which $Q'(t)$ exists and on which Q satisfies the conditions of weak convexity. (See also Notation 3.) Note also that $d\mu_w(t) = v(t)\,dt$ holds only for a.e. $t \in [-1,1]$ and not necessarily for all t for which we define $v(t)$.

Theorem 11. *Let* $w = \exp(-Q)$ *be an admissible weight on* $\mathbb{R}$ *such that* $\min S_w = -1$, $\max S_w = 1$. *Suppose that* Q *is weak convex on* $[-1,1]$ *and that* Q' *is bounded on* $[-1,1]$. *Let* $\kappa \in [-1,1]$ *be a number such that*

$$Q'(x) \leqslant 0 \quad x \in (-1,\kappa),$$

$$Q'(x) \geqslant 0 \quad x \in (\kappa, 1).$$

Then $S_w = [-1,1]$ (*by Theorem* 9) *and the density of* μ_w *is* $d\mu_w(t) = v(t)\,dt$ *a.e.* $t \in [-1,1]$, *where for a.e.* $t \in (-1,\kappa)$

$$\begin{aligned} v(t) &:= \frac{1+t}{\pi^2\sqrt{1-t^2}} \int_{-1}^{1} \frac{(1-s)Q'(s) - (1-t)Q'(t)}{(s-t)\sqrt{1-s^2}}\, ds \\ &\quad + \frac{1}{\pi\sqrt{1-t^2}}\left[1 + \frac{1}{\pi}\int_{-1}^{1} \frac{(1-s)Q'(s)}{\sqrt{1-s^2}}\, ds\right] \end{aligned}$$

and for a.e. $t \in (\kappa, 1)$:

$$v(t) := \frac{1-t}{\pi^2\sqrt{1-t^2}} \int_{-1}^{1} \frac{(1+s)Q'(s)-(1+t)Q'(t)}{(s-t)\sqrt{1-s^2}}\,ds + \frac{1}{\pi\sqrt{1-t^2}}\left[1-\frac{1}{\pi}\int_{-1}^{1}\frac{(1+s)Q'(s)}{\sqrt{1-s^2}}\,ds\right].$$

3. Examples

The following external fields are weak convex on the whole Σ. (We just list external fields which are weak convex, but not convex.)

(1) $Q(x) := \beta x^\alpha$, $\alpha, \beta > 0$ on $\Sigma := [0, +\infty)$. (now $w(x) = \exp(-\beta x^\alpha)$).
(2) $Q(x) := \beta \ln(x+\alpha)$, $\alpha, \beta > 0$ on $\Sigma := [0, \infty)$. (now $w(x) = 1/(x+\alpha)^\beta$).
(3) $Q(x) := 0$ if $x \in [0,1]$ and $Q(x) := \ln x$ if $x \geqslant 1$ on $\Sigma := [0, \infty)$.
(4) $Q(x) := \beta \ln(1+|x|)$ (so $w(x) = 1/(1+|x|)^\beta$) on $\Sigma := [-1, 1]$. More generally, if $\alpha_1, \alpha_2 \geqslant 1$, $\beta_1, \beta_2 \geqslant 0$, $\gamma \in (-1, 1)$ and

$$w(x) := \begin{cases} \dfrac{1}{(\alpha_1 - x)^{\beta_1}} & \text{if } x \in [-1, \gamma], \\ \dfrac{1}{(x+\alpha_2)^{\beta_2}} & \text{if } x \in (\gamma, 1] \end{cases}$$

is continuous on $[-1, 1]$, then the corresponding $Q(x) = -\ln w(x)$ external field is weak convex on $[-1, 1]$.
(5) $Q(x) := 1/(-\ln x)^\alpha$, $\alpha > 0$ on $\Sigma := [0, 1]$.

By Theorem 9, in these examples the support of the extremal measure associated with w is a finite interval: $[a, b]$. So by Theorem 5 (and Theorem C), a continuous function f can be uniformly approximated by weighted polynomials of the form $w^n P_n$ if and only if f vanishes outside the support $[a, b]$. In some cases with the help of Theorem 10 we can get the actual values of a and b.

Finally, here is an example to demonstrate the local usage of Theorem 5.

(6) Consider $Q(x) := \sin x$ on $\Sigma := [0, 2]$. It is not a weak convex function on $[0, 2]$, however $\sin x$ is weak convex on $[0, 0.86]$ with basepoints 0 and 2. Thus by Theorem 9, $J := S_w \cap [0, 0.86]$ is a closed interval. So by Theorem 5, any continuous function which vanishes outside J can be uniformly approximated by weighted polynomials of the form $w^n P_n$, where $w(x) = \exp(-\sin x)$. Notice that it is no longer an if and only if statement, secondly we have to make sure that J is not an empty set, since in this case the statement is meaningless.

3.1. Proofs

First we give a sufficient condition for interchanging differentiation and integration in a parametric integral, which we will need later. This problem is discussed in many calculus books, but here we do not assume that the integrand $F(x,t)$ is a differentiable function of x (when t is fixed), we merely assume absolute continuity. Our belief is that this more general problem should be discussed in books dealing with absolutely continuous functions, in the section where they discuss other classical problems like integration by substitution and integration by parts for absolutely continuous functions. The condition and the proof we give matches the simplicity and usefulness of these other two classical theorems.

Definition 12. Let f be an integrable function. We say that a point $x_0 \in \mathbb{R}$ is a weak Lebesgue point of f, if

$$\lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^{x} f(t)\, dt = f(x_0).$$

We say that a point $x_0 \in \mathbb{R}$ is a Lebesgue point of f, if

$$\lim_{x \to x_0} \frac{1}{x - x_0} \int_{x_0}^{x} |f(t) - f(x_0)|\, dt = 0.$$

Clearly every Lebesgue point is also a weak Lebesgue point. It is known that for an integrable function almost every point is a Lebesgue point, therefore almost every point is a weak Lebesgue point.

Lemma 13. *Let $F : [0,1] \times [0,1] \to \mathbb{R}$ be a function so that $x \mapsto F(x,t_0)$ is absolutely continuous for a.e. $t_0 \in [0,1]$. Thus for a.e. $t_0 \in [0,1]$, $F_1(x,t_0) := \frac{\partial}{\partial x} F(x,t_0)$ exists for a.e. $x \in [0,1]$. Suppose further that $F_1(x,t)$ is measurable on the product space $[0,1] \times [0,1]$. Let $V(t_0) := \int_0^1 |F_1(x,t_0)|\, dx$ (a.e. $t_0 \in [0,1]$) be the total variation of $F_1(x,t_0)$ and $\phi(x) := \int_0^1 F_1(x,t)\, dt$ (a.e. $x \in [0,1]$).*

If $\int_0^1 V(t)\, dt < \infty$ and $x_0 \in [0,1]$ is a weak Lebesgue point of ϕ, then

$$\frac{d}{dx}\left[\int_0^1 F(x,t)\, dt\right](x_0) = \int_0^1 F_1(x_0,t)\, dt.$$

Proof. Using the absolute continuity assumption and the Fubini theorem we get

$$\frac{1}{x_1 - x_0} \int_0^1 (F(x_1,t) - F(x_0,t))\, dt = \frac{1}{x_1 - x_0} \int_0^1 \int_{x_0}^{x_1} F_1(x,t)\, dx\, dt$$
$$= \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} \phi(x)\, dx.$$

(We could change the order of integration since $\int_0^1 \int_0^1 |F_1(x,t)|\, dx\, dt < +\infty$.) But x_0 is a weak Lebesgue point of ϕ, so

$$\lim_{x_1 \to x_0} \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} \phi(x)\, dx = \phi(x_0) = \int_0^1 F_1(x_0, t)\, dt, \tag{4}$$

which proves the statement. $\square$

Corollary 14. *If* $x \mapsto F(x, t_0)$ *is absolutely continuous for a.e.* $t_0 \in [0,1]$ *and* $F_1(x,t) \in L^1([0,1] \times [0,1])$, *then*

$$\frac{d}{dx}\left[\int_0^1 F(x,t)\, dt\right](x_0) = \int_0^1 F_1(x_0, t)\, dt$$

holds for a.e. $x_0 \in [0,1]$.

Proof. $\int_0^1 |\phi(x)|\, dx \leqslant \int_0^1 \int_0^1 |F_1(x,t)|\, dt\, dx < \infty$, so $\phi \in L^1[0,1]$. Thus, almost every point of $[0,1]$ is a weak Lebesgue point of ϕ and the statement follows by applying Lemma 13. $\square$

Proof of Theorem 9. Since S_w is bounded, if any of A and B is infinite we can replace them by a finite value so that $S_w \subset [A, B]$ still holds (see also Proposition 4). So we can assume that A and B are finite.

Suppose indirectly that there exist $a, b \in S_w \cap I$, $a < b$: $(a,b) \cap S_w = \emptyset$.

Let $\mu := \mu_w$ denote the equilibrium measure associated with w and

$$U(x) := U^\mu(x) := \int_{\mathbb{R}} \ln \frac{1}{|x-t|}\, d\mu(t)$$

be the logarithmic potential function of μ. Clearly $U(x)$ is absolutely continuous on every closed subset of (a,b), and because of the Lebesgue monotone convergence theorem, $U(x)$ is continuous on $[a,b]$. (Indeed, we may assume that $|B - A| \leqslant 1$, so $\ln(1/|x-t|) > 0$ $x, t \in [A, B]$. We split the above integral to two integrals, one with measure $\mu|_{(-\infty, a]}$ and the other with measure $\mu|_{[b, +\infty)}$. Since $U(a)$ and $U(b)$ are finite from (1), we can apply Lebesgue's theorem to the two integrals.) So $U(x)$ is absolutely continuous on $[a,b]$.

Since $\int_{\mathbb{R}} \frac{1}{x-t}\, d\mu(t)$ is continuous in (a,b), by Lemma 13,

$$U'(x) = \int \frac{-1}{x-t}\, d\mu(t), \quad x \in (a,b).$$

Notice that both $\frac{-(B-x)}{x-t}$ and $\frac{-(x-A)}{x-t}$ are strictly increasing functions of $x \in (a,b)$ for any fixed $t \in (A, a] \cup [b, B)$ and they are increasing functions if $t = A$ or $t = B$. We know that $\mu(\{A\}) = \mu(\{B\}) = 0$ (because μ has finite logarithmic energy) and since

$S_w \subset [A, B]$ we get that both $(B - x)U'(x)$ and $(x - A)U'(x)$ are strictly increasing on (a, b).

Let $R(x) := U(x) + Q(x)$ and let $\kappa \in [A, B]$ be the number from the definition of weak convexity, that is

$$Q'(x) \leqslant 0 \ \ x \in (A, \kappa) \cap I,$$

$$Q'(x) \geqslant 0 \ \ x \in (\kappa, B) \cap I. \tag{5}$$

Let $E := (B - \kappa)/(\kappa - A)$ if $\kappa \neq A$ and $E := 1$ otherwise. Consider the function

$$f(x) := \begin{cases} (B - x)[U'(x) + Q'(x)] & \text{if } \ x \in (A, \kappa) \cap (a, b) \\ E(x - A)[U'(x) + Q'(x)] & \text{if } \ x \in (\kappa, B) \cap (a, b). \end{cases}$$

Since U' is continuous on (a, b), from (5) it follows that f is a strictly increasing function on the whole (a, b). Therefore, we cannot find numbers $x_1, x_2 \in I, \ x_1 < x_2$ for which both $0 < R'(x_1)$ and $0 > R'(x_2)$ hold.

From Theorem A we have $R(x) := U(x) + Q(x) \geqslant F_w \ x \in (a, b)$ and $R(a), R(b) \leqslant F_w$. It is impossible that $R(x) = F_w$ for all $y \in (a, b)$, because then $f(x)$ would not be strictly increasing. So there is a $y \in (a, b)$: $F_w < R(y)$. If we also use the limit condition of weak convexity (Definition 2), we get

$$0 < R(y) - F_w \leqslant R(y) - R(a) \leqslant R(y) - \lim_{x \to a+0} R(x) = \lim_{x \to a+0} \int_x^y R'(t)\, dt,$$

which implies the existence of $x_1 \in (a, y)$: $0 < R'(x_1)$. Similarly

$$\begin{aligned} 0 > F_w - R(y) \geqslant R(b) - R(y) &\geqslant \lim_{x \to b-0} R(x) - R(y) \\ &= \lim_{x \to b-0} \int_y^x R'(t)\, dt, \end{aligned}$$

so there is an $x_2 \in (y, b)$: $0 > R'(x_2)$. This is a contradiction. $\square$

Proof of Theorem 10. As in the proof of Theorem 9, we can assume again that A and B are finite.

We shall only prove (i) and (iii), the other two follow by the symmetry of the statement.

If $K \subset \mathbb{R}$ is a compact set, we define

$$F(K) := \log \operatorname{cap}(K) - \int Q\, d\omega_K$$

where ω_K is the equilibrium measure of K. This is called the F-functional for w and from [4, Theorem IV.1.5], we know that

$$F(S_w) = F([a, b]) = \max_{\alpha, \beta} F([\alpha, \beta]), \tag{6}$$

where the maximum is taken over all nondegenerate intervals $[\alpha, \beta] \subset [A, B]$. Now from [4, Example I.3.5] we have

$$\mathrm{cap}[\alpha, \beta] = \frac{\beta - \alpha}{4}, \quad d\omega_{[\alpha,\beta]} = \frac{dx}{\pi\sqrt{(x-\alpha)(\beta - x)}}, \quad x \in [\alpha, \beta]$$

and so

$$F([\alpha, \beta]) = \log \frac{\beta - \alpha}{4} - \frac{1}{\pi}\int_\alpha^\beta \frac{Q(x)}{\sqrt{(x-\alpha)(\beta-x)}}\,dx.$$

On making the change of variable

$$x = \frac{\beta + a}{2} + \frac{\beta - a}{2}\cos\theta, \quad 0 \leqslant \theta \leqslant \pi,$$

we can rewrite $F([a, \beta])$ as

$$F([a, \beta]) = \log \frac{\beta - a}{4} - \frac{1}{\pi}\int_0^\pi Q\left(\frac{\beta + a}{2} + \frac{\beta - a}{2}\cos\theta\right) d\theta.$$

Using (1) and the lower semi-continuity of Q it follows that Q is bounded on $[a, b]$. Thus from the Lebesgue dominated convergence theorem $\beta \mapsto F([a, \beta])$ is a continuous function on $[a, b]$. Interchanging differentiation and integration, we get

$$\begin{aligned}\frac{\partial}{\partial\beta}F([a, \beta]) = {}& \frac{1}{\beta - a} - \frac{1}{2\pi}\int_0^\pi Q'\left(\frac{\beta + a}{2} + \frac{\beta - a}{2}\cos\theta\right) \\ & \times (1 + \cos\theta)\,d\theta \quad \beta \in (a, B).\end{aligned} \tag{7}$$

To verify the differentiation, we will show that

$$\begin{aligned}h(\beta) &:= \int_0^\pi Q'\left(\frac{\beta + a}{2} + \frac{\beta - a}{2}\cos\theta\right)(1 + \cos\theta)\,d\theta \\ &= \frac{2}{\beta - a}\int_a^\beta \sqrt{\frac{x - a}{\beta - x}}\,Q'(x)\,dx\end{aligned}$$

is a continuous real function of β on (a, B) and

$$\beta \mapsto \int_0^\pi \left| Q'\left(\frac{\beta + a}{2} + \frac{\beta - a}{2}\cos\theta\right)(1 + \cos\theta)\right| d\theta \tag{8}$$

is bounded inside (a, B) (see Lemma 13). We will also see that $\lim_{\beta\to B^-} h(\beta)$ exists and equal to $h(B)$ (infinity value is allowed). Thus from the Lagrange mean value theorem we can conclude that (7) holds on $(a, B]$ in the case when $b = B$. Now if $b<B$, we gain (i) immediately from (7), since by (6) $\frac{\partial}{\partial\beta}F([a,\beta])$ has to be zero at $\beta = b$. On the other hand if $b = B$, we gain (iii) by the same logic, since by (6) the left derivative has to be $\frac{\partial}{\partial\beta}F([a,\beta])|_{\beta=B}\leqslant 0$.

The boundedness of (8) can be proved similarly as the finiteness of $h(\beta)$ on (a, B) (see below). Thus it remained to prove that $h(\beta)$ is continuous on $(a, B]$ in the extended sense. Let κ be a number so that

$$Q'(x)\leqslant 0 \quad x\in[A,\kappa)$$

$$Q'(x)\geqslant 0 \quad x\in(\kappa,B].$$

First we show that $h(\beta)$ is a finite valued function on (a, B). So let $\beta\in(a,B)$. Notice that because of our monotonicity assumptions, Q' is bounded inside (A, B). So $h(\beta)$ is clearly finite if $A<a$. If, however, $a = A$, we have to distinguish two cases:

- If $A<\kappa$, then

$$\int_A^\kappa Q'(x)\,dx = \lim_{\varepsilon\to 0^+}\int_{A+\varepsilon}^\kappa Q'(x)\,dx = \lim_{\varepsilon\to 0^+}(Q(\kappa)-Q(A+\varepsilon))> -\infty,$$

  since Q is bounded on $[a, b]$.
- If $A = \kappa$, then

$$h(\beta) = \frac{2}{\beta-a}\int_A^\beta \frac{(x-A)Q'(x)}{\sqrt{(x-A)(\beta-x)}}dx< +\infty,$$

  since $(x-A)Q'(x)$ is a (non-negative) increasing function.

These show that $h(\beta)$ is finite in both cases.

We are done, if we can show that for any $u\in[A,B)$ and for any $f\in L^1[A,B]$ non-negative decreasing function,

$$\int_u^\beta \frac{f(x)\sqrt{x-a}}{(B-x)\sqrt{\beta-x}}dx$$

is a continuous function of β on $(u, B]$, while if $g\in L^1[A,B]$ is any non-negative increasing function, then

$$\int_u^\beta \frac{g(x)\sqrt{x-a}}{(x-A)\sqrt{\beta-x}}dx \tag{9}$$

is continuous on $(u, B]$ and continuous on $[u, B]$, if $A<u$.

Indeed, if $a<\beta\leqslant\kappa$, then we use

$$\int_a^\beta \sqrt{\frac{x-a}{\beta-x}}\,Q'(x)\,dx = -\int_a^\beta \frac{f(x)\sqrt{x-a}}{(B-x)\sqrt{\beta-x}}\,dx,$$

where $f(x) := -(B-x)Q'(x)$ is decreasing on $[A,\kappa)$, while if $a<\kappa\leqslant\beta\leqslant B$ or $\kappa\leqslant a<\beta\leqslant B$, we use

$$\int_a^\beta \sqrt{\frac{x-a}{\beta-x}}\,Q'(x)\,dx = \int_a^\kappa \sqrt{\frac{x-a}{\beta-x}}\,Q'(x)\,dx + \int_\kappa^\beta \frac{g(x)\sqrt{x-a}}{(x-A)\sqrt{\beta-x}}\,dx$$

or

$$\int_a^\beta \sqrt{\frac{x-a}{\beta-x}}\,Q'(x)\,dx = \int_a^\beta \frac{g(x)\sqrt{x-a}}{(x-A)\sqrt{\beta-x}}\,dx,$$

respectively, where $g(x) := (x-A)Q'(x)$ is increasing on $(\kappa,B]$. Here

$$\int_a^\kappa \sqrt{\frac{x-a}{\beta-x}}\,Q'(x)\,dx$$

is clearly a continuous function of β on $[\kappa,B]$ by the Lebesgue monotone convergence theorem.

Consider

$$\begin{aligned} h_1(\beta) &:= \int_u^\beta \frac{f(x)\sqrt{x-a}}{(B-x)\sqrt{\beta-x}}\,dx \\ &= \frac{\beta-u}{2}\int_0^\pi \frac{f(\frac{\beta+u}{2}+\frac{\beta-u}{2}\cos\theta)(1+\cos\theta)}{(B-(\frac{\beta+u}{2}+\frac{\beta-u}{2}\cos\theta))}\,d\theta. \end{aligned}$$

If $\beta\to\beta_0\in(u,B)$, the integrands at the right-hand side have an integrable majorant $(Cf(\frac{\beta_1+u}{2}+\frac{\beta_1-u}{2}\cos\theta)$, where $u<\beta_1<\beta_0)$, so by the Lebesgue dominated convergence theorem

$$\lim_{\beta\to\beta_0} h_1(\beta) = \frac{\beta_0-u}{2}\int_0^\pi \frac{\lim_{\beta\to\beta_0} f(\frac{\beta+u}{2}+\frac{\beta-u}{2}\cos\theta)(1+\cos\theta)}{(B-(\frac{\beta_0+u}{2}+\frac{\beta_0-u}{2}\cos\theta))}\,d\theta.$$

But f is continuous almost everywhere and $\arccos x$ is an absolutely continuous function, thus for almost every $\theta\in[0,\pi]$ we have $\lim_{\beta\to\beta_0} f(\frac{\beta+u}{2}+\frac{\beta-u}{2}\cos\theta) = f(\frac{\beta_0+u}{2}+\frac{\beta_0-u}{2}\cos\theta)$. This means that $h_1(\beta)$ is continuous at β_0.

Now let $\beta_0 := B$ and we will prove that $h_1(\beta)$ is continuous at B from the left in the extended sense (i.e., $h_1(B) = +\infty$ is allowed). We may suppose right away that $h_1(B) < +\infty$, since if $h_1(B) = +\infty$, then $\lim_{\beta\to B^-} h_1(\beta) = +\infty$ is clear.

Let $0 < D$ be any (big) number and $\varepsilon := \varepsilon(\beta) := B - \beta$. Consider

$$h_1(\beta) = \int_u^\beta \frac{f(x)\sqrt{x-a}}{(B-x)\sqrt{\beta - x}}\, dx = \int_u^{\beta - D\varepsilon} + \int_{\beta - D\varepsilon}^{\beta}. \tag{10}$$

Now

$$\begin{aligned} \int_u^{\beta - D\varepsilon} &\leqslant \sqrt{\frac{B-(\beta - D\varepsilon)}{\beta - (\beta - D\varepsilon)}} \int_u^{\beta - D\varepsilon} \frac{f(x)\sqrt{x-a}}{(B-x)^{1.5}}\, dx \\ &\leqslant \sqrt{\frac{D+1}{D}} \int_u^B \frac{f(x)\sqrt{x-a}}{(B-x)^{1.5}}\, dx. \end{aligned} \tag{11}$$

We will show that the second term in (10), $\int_{\beta - D\varepsilon}^{\beta}$ is going to 0, as $\beta \to B^-$. Since $0 < D$ was arbitrary, this together with (11) imply that

$$\limsup_{\beta \to B^-} h_1(\beta) \leqslant h_1(B). \tag{12}$$

Now

$$\begin{aligned} \int_{\beta - D\varepsilon}^{\beta} \frac{f(x)\sqrt{x-a}}{(B-x)\sqrt{\beta - x}}\, dx &\leqslant C \frac{f(\beta - D\varepsilon)}{B - \beta} \int_{\beta - D\varepsilon}^{\beta} \frac{1}{\sqrt{\beta - x}}\, dx \\ &\leqslant C_D \frac{f(\beta - D\varepsilon)}{\sqrt{B - (\beta - D\varepsilon)}}, \end{aligned} \tag{13}$$

where C_D depends on D, but not on β. As $\beta \to B^-$, $\varepsilon = \varepsilon(\beta) \to 0$, so to prove that the right-hand side of (13) is going to 0, we have to show that

$$\lim_{\beta \to B^-} \frac{f(\beta)}{\sqrt{B - \beta}} = 0.$$

This latter limit follows from

$$h_1(B) = \int_u^B \frac{f(x)\sqrt{x-a}}{(B-x)^{1.5}}\, dx < +\infty.$$

Indeed, with a simplified notation, we claim that

$$\lim_{\varepsilon \to 0^+} \frac{f(\varepsilon)}{\sqrt{\varepsilon}} = 0 \tag{14}$$

follows from $\int_0^1 f(x)\sqrt{d-x}/x^{1.5}\, dx < \infty$ (where f is any non-negative increasing function and $1 \leqslant d$ is arbitrary).

Notice that for any small $0<\varepsilon$,

$$\int_{\varepsilon}^{2\varepsilon} \frac{f(x)\sqrt{d-x}}{x^{1.5}}\,dx \geqslant C\frac{f(\sqrt{\varepsilon})}{\sqrt{\varepsilon}},$$

so if we choose any $\varepsilon_1>\varepsilon_2>\varepsilon_3>\cdots>0$ with the constraint that $[\varepsilon_1, 2\varepsilon_1]$, $[\varepsilon_2, 2\varepsilon_2], \ldots$ are disjoint intervals, then

$$\infty > \int_0^1 \frac{f(x)\sqrt{d-x}}{x^{1.5}}\,dx \geqslant \sum_{i=1}^{\infty} \int_{\varepsilon_i}^{2\varepsilon_i} \frac{f(x)\sqrt{d-x}}{x^{1.5}}\,dx \geqslant C\sum_{i=1}^{\infty}\frac{f(\varepsilon_i)}{\sqrt{\varepsilon_i}},$$

which implies that $\lim(f(\varepsilon_i)/\sqrt{\varepsilon_i}) = 0$. From this fact (14) follows easily.

So (12) is proved. To get

$$\liminf_{\beta\to B^-}\ h_1(\beta) \geqslant h_1(B)$$

we just have to look at

$$\int_u^{\beta} \frac{f(x)\sqrt{x-a}}{(B-x)\sqrt{\beta-x}}\,dx \geqslant \int_u^{\beta} \frac{f(x)\sqrt{x-a}}{(B-x)^{1.5}}\,dx$$
$$\to \int_u^{B} \frac{f(x)\sqrt{x-a}}{(B-x)^{1.5}}\,dx \quad \text{as } \beta\to B^-.$$

Therefore the continuity of $h_1(\beta)$ at B is established.

The continuity of (9) on $(u, B]$ follows with the same argument. Finally, if $A<u$, then $g(x)$ and $x-A$ are bounded in a neighborhood of u, so (9) tends to 0 as β tends to u from the left, hence (9) is continuous on $[u, B]$.

The proof of Theorem 10 is now complete. $\square$

Definition 15. Let $f\in L^1[a,b]$. For $x\in(a,b)$ the Cauchy principal value integral is defined as

$$PV\int_a^b \frac{f(s)}{s-x} := \lim_{\varepsilon\to 0}\left(\int_a^{x-\varepsilon} \frac{f(s)}{s-x}\,ds + \int_{x+\varepsilon}^b \frac{f(s)}{s-x}\,ds\right)$$

if this limit exists. If $x\notin(a,b)$, the PV integral is simply an ordinary Lebesgue integral (which clearly exists if $x\notin[a,b]$):

$$PV\int_a^b \frac{f(s)}{s-x}\,ds := \int_a^b \frac{f(s)}{s-x}\,ds.$$

A well-known theorem states that for almost every x in $[-1,1]$ the above principal value integral exists and finite.

Proof of Theorem 11

Lemma 16. *Let* $w = \exp(-Q)$ *be an admissible weight on* $\mathbb{R}$ *so that* $S_w = [-1, 1]$. *Suppose that* Q *is absolutely continuous on* $[-1, 1]$ *and that* Q' (*which exists a.e. in* $[-1, 1]$) *is bounded in* $[-1, 1]$. *Let*

$$v(t) := \frac{1}{\pi^2\sqrt{1-t^2}} PV \int_{-1}^{1} \frac{\sqrt{1-s^2}Q'(s)}{s-t} ds + \frac{1}{\pi\sqrt{1-t^2}}, \quad t \in [-1, 1]. \tag{15}$$

If $0 \leqslant v(t)$ *a.e.* $t \in [-1, 1]$, *then* $d\mu_w(t) = v(t)\, dt$ *a.e.* $t \in [-1, 1]$.

Proof. Let $w_1(x) = \exp(-Q_1(x))$ be and admissible weight on $\mathbb{R}$ such that Q_1 is absolutely continuous on $[0, 1]$ and Q_1' is bounded on $[0, 1]$. Consider the expression

$$g(t) := \frac{1}{\pi^2}\sqrt{\frac{1-t}{t}} PV \int_0^1 \frac{sQ_1'(s)}{(s-t)\sqrt{s(1-s)}} ds + \frac{1}{\pi\sqrt{t(1-t)}}\left(1 - \frac{1}{\pi}\int_0^1 \sqrt{\frac{s}{1-s}} Q_1'(s)\, ds\right).$$

Exactly as in the proof of Theorem IV.3.2. in [4], if we set $f(x) = Q_1(x^2)/2$ and apply Theorem IV.3.1 in [4], we get that $\int_0^1 g = 1$ and $\int_0^1 \ln|x-t| g(t)\, dt = Q_1(x) + C$ with some constant C for all $x \in (-1, 1)$. ([4, Theorem IV.3.1], is originated from [3]. Note that in the formulation of this theorem there is a missing hypothesis: f should be absolutely continuous on $[-1, 1]$.) If we transfer this statement from $[0, 1]$ to $[-1, 1]$ by a linear transformation, we get the following:

Let

$$v(t) := \frac{1-t}{\pi^2\sqrt{1-t^2}} PV \int_{-1}^{1} \frac{(1+s)Q'(s)}{(s-t)\sqrt{1-s^2}} ds + \frac{1}{\pi\sqrt{1-t^2}}\left(1 - \frac{1}{\pi}\int_{-1}^{1} \frac{(1+s)Q'(s)}{\sqrt{1-s^2}} ds\right) \quad t \in (-1, 1),$$

then $\int_{-1}^{1} v = 1$, and

$$\int_{-1}^{1} \ln|x-t| v(t)\, dt = Q(x) + C \quad x \in (-1, 1) \tag{16}$$

with some constant C. We assumed that $0 \leqslant v(t)$ almost everywhere and therefore by Theorem I.3.3 in [4], $d\mu_w(t) = v(t)\, dt$ a.e. $t \in [-1, 1]$ as we stated. (Because of (16) and the boundedness of Q on $[-1,1]$, the finite logarithmic energy condition of I.3.3 is satisfied.)

To get (15) we just have to combine the two integrals above in the representation of $v(t)$. $\square$

Now we can prove Theorem 11 as follows:

Let $r \in [-1,1]$ be arbitrary. We will find another form of the function v in Lemma 16. We will make use of the identity (see [4, Formula IV (3.20)])

$$PV \int_{-1}^{1} \frac{1}{(s-t)\sqrt{1-s^2}} ds = 0 \quad t \in (-1,1). \text{ Now} \tag{17}$$

$$\begin{aligned} v(t) =& \frac{1}{\pi^2\sqrt{1-t^2}} \left[PV \int_{-1}^{r} \frac{(1-s^2)Q'(s)}{(s-t)\sqrt{1-s^2}} ds \right. \\ & \left. + PV \int_{r}^{1} \frac{(1-s^2)Q'(s)}{(s-t)\sqrt{1-s^2}} ds \right] + \frac{1}{\pi\sqrt{1-t^2}} \\ =& \frac{1}{\pi^2\sqrt{1-t^2}} \left[PV \int_{-1}^{r} \frac{[(1+t)(1-s)+(s-t)(1-s)]Q'(s)}{(s-t)\sqrt{1-s^2}} ds \right. \\ & \left. + PV \int_{r}^{1} \frac{[(1-t)(1+s)-(s-t)(1+s)]Q'(s)}{(s-t)\sqrt{1-s^2}} ds \right] + \frac{1}{\pi\sqrt{1-t^2}}, \end{aligned}$$

so

$$\begin{aligned} v(t) =& \frac{1+t}{\pi^2\sqrt{1-t^2}} PV \int_{-1}^{r} \frac{(1-s)Q'(s)}{(s-t)\sqrt{1-s^2}} ds \\ & + \frac{1-t}{\pi^2\sqrt{1-t^2}} PV \int_{r}^{1} \frac{(1+s)Q'(s)}{(s-t)\sqrt{1-s^2}} ds + \frac{1}{\pi\sqrt{1-t^2}} \\ & + \frac{1}{\pi^2\sqrt{1-t^2}} \left[\int_{-1}^{r} \frac{(1-s)Q'(s)}{\sqrt{1-s^2}} ds - \int_{r}^{1} \frac{(1+s)Q'(s)}{\sqrt{1-s^2}} ds \right]. \end{aligned} \tag{18}$$

Let $\kappa \in [-1,1]$ be a number so that

$$Q'(x) \leqslant 0 \quad x \in (-1,\kappa)$$

$$Q'(x) \geqslant 0 \quad x \in (\kappa,1).$$

Let $t \in (-1, \kappa)$ be a value for which $Q'(t)$ exists. Setting $r := 1$ and using (17), we get

$$v(t) = \frac{1+t}{\pi^2\sqrt{1-t^2}} PV \int_{-1}^{1} \frac{(1-s)Q'(s) - (1-t)Q'(t)}{(s-t)\sqrt{1-s^2}} ds + \frac{1}{\pi\sqrt{1-t^2}}\left[1 + \frac{1}{\pi}\int_{-1}^{1} \frac{(1-s)Q'(s)}{\sqrt{1-s^2}} ds\right].$$

By Theorem 10 the second term is non-negative.

From the assumptions of Theorem 11, clearly

$$\frac{(1-s)Q'(s) - (1-t)Q'(t)}{s-t}$$

is non-negative, when $s \in (-1, \kappa)$, and it is also non-negative, when $s \in (\kappa, 1)$. (For the latter, consider the sign of Q'.)

Thus on the set $\{t \in (-1, \kappa)\colon\ Q'(t)\ exists\}$ we have $0 \leqslant v(t)$ and $v(t)$ has the form as given in Theorem 11. If in (18) we set $r := -1$, we can see the same way that on the set $\{t \in (\kappa, 1)\colon\ Q'(t)\ exists\}$ we have $0 \leqslant v(t)$ and $v(t)$ has the form as given in Theorem 11.

So $0 \leqslant v(t)$ *a.e.* $t \in [-1, 1]$, and $d\mu_w(t) = v(t)\, dt$ follows from the previous lemma. $\square$

Proof of Theorem 6. Since w is absolutely continuous inside (a, b) and $dw(x)/dx = -w(x)Q'(x)$ is in every L^p, $1 < p < \infty$ inside (a, b) (because Q' is bounded inside (a, b)), it follows from [4, Theorem IV.2.2] that μ_Q is absolutely continuous and its derivative is in every L^p inside (a, b). We shall denote the density $d\mu_Q(t)/dt$ by $v(t)$.

Now we prove the first part of Theorem 6, that is:

Lemma 17. *If $(a, b) \subset S_w$ and Q is weak convex on (a, b) with basepoints* $\min S_w$ *and* $\max S_w$, *then v has a positive lower bound inside (a, b).*

Proof. The same argument works as in the proof of Theorem F (see [6]), since now Theorem 9 is at our disposal.

For any positive λ, clearly λQ is also weak convex on (a, b) with basepoints $\min S_w$ and $\max S_w$, so by Theorem 9, $S_{w^\lambda} \cap (a, b)$ is an interval with endpoints $a_\lambda,\ b_\lambda$. We show that if $1 < \lambda$ is sufficiently close to 1, then a_λ is sufficiently close to a and b_λ is sufficiently close to b.

It is enough to prove that in any neighborhood of any point x_0 of (a, b) there is a point x_1 lying in some (a_λ, b_λ), $1 < \lambda$. Indeed, then this property and the decreasing character of the support S, namely $S_{w^{r_1}} \subset S_{w^{r_2}}$ for $r_1 > r_2$ (see [4, Theorem IV.4.1]) imply our claim.

Now we use the following characterization of points in the support S_w: $x_0 \in S_w$ if and only if for every neighborhood B of x_0, there is an n and a polynomial P_n of degree at most n such that $w^n|P_n|$ attains its maximum in Σ at some point of $B \cap \Sigma$ and nowhere outside B (see [4, Corollary IV.1.4]). By continuity then the same is true of $w^{\lambda n}|P_n|$ for some $1 < \lambda$ sufficiently close to 1. But $||w^{\lambda n} P_n||_\Sigma = ||w^{\lambda n} P_n||_{S_{w^\lambda}}$. ($\mathbf{C} \setminus \Sigma$ is a regular domain, so for any continuous w we have $||w^n P_n||_\Sigma = ||w^n P_n||_{S_w}$, see [4, Corollary III.2.6]) Therefore, $B \cap S_{w^\lambda} \neq \emptyset$ and so in B there is an x_1 lying in (a_λ, b_λ) as we claimed above.

Thus, if $[a', b']$ is any subinterval of (a, b), there is a $\lambda > 1$ such that $[a', b']$ is in the support of μ_{w^λ}. Now we invoke the inequality ([4, Theorem IV.4.9]):

$$\mu_w|_{S_{w^\lambda}} \geqslant \frac{1}{\lambda}\mu_{w^\lambda} + \left(1 - \frac{1}{\lambda}\right)\omega_{S_w}|_{S_{w^\lambda}},$$

where ω_{S_w} denotes the equilibrium measure of the set S_w (which is nothing else than the equilibrium measure corresponding to the identically zero field on S_w). Since $[a', b']$ is part of S_{w^λ} and the equilibrium measure ω_{S_w} has a positive and continuous density in (a', b'), it follows that the density of μ_w has a positive lower bound inside (a', b'). Here, $[a', b'] \subset (a, b)$ was arbitrary, so the proof of the lemma is complete. $\square$

It remained to prove the second part of Theorem 6. To do this, first we need the concept of the balayage measure. Let v be a measure on the real line and K be an interval. There is a unique measure $\bar{v}$ supported on K such that the total mass of $\bar{v}$ equals the total mass of v and for some constant d we have $U^{\bar{v}}(x) = U^v(x) + d$ for every $x \in K$. $\bar{v}$ is called the balayage of v onto K. Actually, the balayage process moves (sweeps) only the part of v lying outside K, i.e.,

$$\bar{v} = v|_K + \overline{v|_{\mathbb{R} \setminus K}}. \tag{19}$$

For the second measure on the right there is a closed form (see [4, Formula II.4.47]), which shows that by taking balayage onto K, we add to the portion of v lying in K a measure with a continuous density.

The relevance of the balayage to extremal fields is explained by the following: if $K \subset S_w$ is a closed interval and w_1 is the restriction of w onto K (i.e., the weight w_1 is considered on K), then the equilibrium measure μ_{w_1} associated with w_1 is the balayage of μ_w onto K (see [4, Theorem IV.1.6(e)]).

Now we will make some elementary observations regarding functions with "smooth integral". We leave the proofs of the first two propositions to the reader.

We say that a family of functions has uniformly smooth integrals, if the δ in the definition of a function with smooth integral is independent of the function in the family. (See also Definition D.)

Proposition 18. *Non-negative linear combination of finitely many functions with smooth integrals is again with smooth integral. More generally, if v is a finite positive Borel measure on $B \subset \mathbb{R}$ and $\{v_s(x): s \in B\}$ is a family of functions with uniformly smooth* *integral on* $[a,b]$ *for which* $s \mapsto v_s(x)$ *is a measureable function for a.e.* $x \in [a,b]$, *then*

$$v(x) := \int_B v_s(x)\, d\nu(s)$$

has also smooth integral on $[a,b]$.

The proof of the next simple statement can be found in [6].

Proposition 19. *If* v_1 *has smooth integral,* $0 \leqslant v_2$ *is continuous, and* $v_1 - v_2$ *has a positive lower bound, then* $v_1 - v_2$ *has also smooth integral.*

Proposition 20. *The function* $\log(1/|x|)$ *has smooth integral on any interval* $[-a,a]$, *where* $0 < a < 1$.

Proof. Let

$$R_{b,\tau} := \frac{\int_{b-\tau}^{b} -\ln|x|\,dx}{\int_{b}^{b+\tau} -\ln|x|\,dx},$$

where $-a \leqslant b-\tau < b+\tau \leqslant a$. We have to show that for any $0<\varepsilon$ there is a $0<\delta$ such that $|R_{b,\tau}-1| \leqslant \varepsilon$ whenever $\tau \leqslant \delta$.

By symmetry we can suppose that $0 \leqslant b$.

Suppose first that $b \leqslant 2\tau$. Obviously,

$$1 \leqslant R_{b,\tau} \leqslant \frac{\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} -\ln|x|\,dx}{\int_{2\tau}^{3\tau} -\ln|x|\,dx} = \frac{\tau(1-\ln\frac{\tau}{2})}{3\tau(1-\ln(3\tau)) - 2\tau(1-\ln(2\tau))}$$
$$= \frac{1-\ln\frac{\tau}{2}}{1+\ln(16/216)-\ln\frac{\tau}{2}},$$

which tends to 1 as $\tau \to 0$.

Therefore from now on we can suppose that $2\tau \leqslant b$. Again $1 \leqslant R_{b,\tau}$. Let us suppose indirectly that for some $0<\varepsilon$ there are positive sequences b_n and τ_n such that $[b_n - \tau_n, b_n + \tau_n] \subset [\tau_n, a]$, $\tau_n \to 0$, and $1+\varepsilon \leqslant R_{b_n,\tau_n}$.

Since τ_n/b_n is bounded, we may select a converging subsequence from it, so we can assume that $\tau_n/b_n \to \rho$, where $\rho \in [0, \frac{1}{2}]$. By direct calculation

$$\begin{aligned} R_{b_n,\tau_n} &= \frac{\tau_n - b_n \ln b_n + (b_n-\tau_n)\ln(b_n-\tau_n)}{\tau_n + b_n\ln b_n - (b_n+\tau_n)\ln(b_n+\tau_n)} \\ &= \frac{\tau_n(1-\ln b_n) + (b_n-\tau_n)\ln(1-\frac{\tau_n}{b_n})}{\tau_n(1-\ln b_n) - (b_n+\tau_n)\ln(1+\frac{\tau_n}{b_n})} \\ &= \frac{1 - \frac{\frac{1+b_n}{\tau_n}\ln(1-\frac{\tau_n}{b_n}) - \ln(1-\frac{\tau_n}{b_n})}{\ln b_n}}{1 - \frac{1-\frac{b_n}{\tau_n}\ln(1+\frac{\tau_n}{b_n}) - \ln(1+\frac{\tau_n}{b_n})}{\ln b_n}}. \end{aligned} \tag{20}$$

If $0<\rho$, then $b_n \to 0$ necessarily, thus $\ln b_n \to -\infty$ and the limit of (20) is 1.

If, however, $\rho = 0$, then we use the elementary limits $\lim_{\gamma\to 0} \frac{1}{\gamma}\ln(1-\gamma) = -1$ and $\lim_{\gamma\to 0} \frac{1}{\gamma}\ln(1+\gamma) = 1$. Thus $\lim(1 + \frac{b_n}{\tau_n}\ln(1-\frac{\tau_n}{b_n}) - \ln(1-\frac{\tau_n}{b_n})) = 0$ and $\lim(1 - \frac{b_n}{\tau_n}\ln(1+\frac{\tau_n}{b_n}) - \ln(1+\frac{\tau_n}{b_n})) = 0$ (as $n\to\infty$), which imply that the limit of (20) is again 1. This contradiction proves the lemma. $\square$

After these preparations we start proving the second part of Theorem 6, that is:

Theorem 21. *If* $(a, b) \subset S_w$ *and* Q *is weak convex on* (a, b) *with basepoints* $\min S_w$ *and* $\max S_w$, *then* v *has a smooth integral inside* (a, b).

Proof. Let us restrict w to $[a, b]$. Based on what we said about the balayage and smooth integral, it is enough to prove that the equilibrium measure associated with this restricted weight function has a density v_1 which has smooth integral inside (a, b). Indeed, by (19), $v = v_1 - v_2$, where $0 \leqslant v_2$ is continuous and v has a positive lower bound inside (a, b), so if v_1 has smooth integral inside (a, b), so does v.

Therefore from now on we will assume that w is defined on $[a, b]$, i.e., $\Sigma = [a, b]$. We will continue to use v for the density of the equilibrium measure associated with this restricted w. Furthermore, because of the balayage process, the new support S_w is the interval $[a, b]$ (so v is defined on $[a, b]$). As a result of this, we will be able to apply Theorem 11 to get a formula for v. (We remark that the hypothesis of Theorem 21 is still satisfied, since Q is weak convex on (a, b) by Proposition 4.)

Now we will prove three lemmas:

Lemma 22. *Let* $f \in L^1[c, d]$ *be a function and suppose that* $f(s)$ *is Lipschitz continuous on* $[c, \frac{c+d}{2}]$ *with Lipschitz constant L. Let* $v^*(t) := \int_c^d \frac{1}{s-t} f(s)\, ds\ (t < c)$. *Then for every* $t \in (c - \min(\exp\frac{-4}{(d-c)^2}, \frac{1}{e}), c)$

$$\left|\frac{v^*(t)}{-\ln(c-t)} - f(c)\right| \leqslant \frac{(1 + \frac{d-c}{2})(L + |f(c)|) + \|f\|_{L^1[c,d]}}{\sqrt{-\ln(c-t)}}.$$

Proof. Let $t \in (c - \min(\exp\frac{-4}{(d-c)^2}, \frac{1}{e}), c)$. Then $c - t < \frac{1}{e}$ and we can define

$$\tau := \sqrt{\frac{1}{-\ln(c-t)}}.$$

Because of the Lipschitz continuity of f, $|f(s) - f(c)| \leqslant L(s-c)$ for all $s \in [c, (c+d)/2]$. If we divide by $s - t$ and integrate, we get

$$\left|\int_c^{c+\tau} \frac{1}{s-t} f(s)\, ds - \int_c^{c+\tau} \frac{1}{s-t} f(c)\, ds\right|$$
$$\leqslant L\int_c^{c+\tau} \frac{s-c}{s-t}\, ds \leqslant L\tau \int_c^{c+\tau} \frac{1}{s-t}\, ds.$$

(Note that from $t \in (c - \min(\exp\frac{-4}{(d-c)^2}, \frac{1}{e}), c)$ it follows that $c + \tau \leqslant (c+d)/2$.)

This means that

$$\left| v^*(t) - \int_{c+\tau}^{d} \frac{1}{s-t} f(s)\, ds - (\ln(c+\tau-t) - \ln(c-t)) f(c) \right| \\ \leqslant L\tau(\ln(c+\tau-t) - \ln(c-t)),$$

and here $|\int_{c+\tau}^{d}| \leqslant ||f||_{L^1}/\tau$. By the triangle inequality we get

$$\left| \frac{v^*(t)}{-\ln(c-t)} - f(c) \right| \leqslant L\tau\left(1 + \frac{\ln(c+\tau-t)}{-\ln(c-t)}\right) \\ + \frac{|\ln(c+\tau-t)|}{-\ln(c-t)} |f(c)| + \tau ||f||_{L^1}.$$

Notice that if $\ln(c+\tau-t) \geqslant 0$, then

$$\ln((c-t)+\tau) \leqslant \ln\left(1 + \frac{d-c}{2}\right) \leqslant \frac{d-c}{2}$$

because as we mentioned, $\tau \leqslant (d-c)/2$. If however $\ln(c+\tau-t) < 0$, then

$$|\ln(c+\tau-t)| = -\ln((c-t)+\tau) \leqslant -\ln(\tau) \leqslant \frac{1}{\tau}.$$

Therefore in all cases

$$\left| \frac{v^*(t)}{-\ln(c-t)} - f(c) \right| \\ \leqslant L\tau\left(1 + \frac{(d-c)/2}{-\ln(c-t)}\right) + \left(\frac{1/\tau}{-\ln(c-t)} + \frac{(d-c)/2}{-\ln(c-t)}\right)|f(c)| + \tau ||f||_{L^1}.$$

Since from $t \geqslant c - \frac{1}{e}$ we have $\tau^2 \leqslant \tau \leqslant 1$, the statement follows immediately. □

Lemma 23. *Let* $-1 < \alpha < \beta < 1$ *and let* $a(t), b(t), f(t), g(t)$ *be positive continuous functions on* $(-1, 1)$ *so that* $f(t), g(t) \in L^1[-1, 1]$. *Suppose also that* $f(s)$ *is Lipschitz continuous on* $[\alpha, \frac{\beta+1}{2}]$ *and* $g(s)$ *is Lipschitz continuous on* $[\frac{\alpha-1}{2}, \beta]$. *Define*

$$\phi_c(t) := \begin{cases} a(t) \int_c^1 \frac{1}{s-t} f(s)\, ds, & t \in (-1, c), \\ 0, & t \in (c, 1) \end{cases}$$

$$\psi_c(t) := \begin{cases} 0, & t \in (-1, c), \\ b(t) \int_{-1}^{c} \frac{1}{t-s} g(s)\, ds, & t \in (c, 1) \end{cases}$$

and

$$\rho_c(t) := \phi_c(t) + \psi_c(t), \quad \mathscr{F} := \{\rho_c(t) : \ c \in [\alpha, \beta]\}.$$

If $a(c)f(c) = b(c)g(c)$ for all $c \in [\alpha, \beta]$, then the family of function $\mathscr{F}$ has uniformly smooth integral on $[\alpha, \beta]$.

Proof. In this proof let us agree on the following (unusual) notation: we say that a function (of t, c, I and J) is $o(1)$ if it is uniformly tending to 0 on its specified domain as $\varepsilon \to 0$. This domain can depend on ε. For example since $a(t)$ is a continuous function, we may write: $a(t) = a(c) + o(1)$ for $t \in [c, c+\varepsilon]$ as $\varepsilon \to 0$.

Let $I := [u-\varepsilon, u]$ and $J := [u, u+\varepsilon]$ be two adjacent intervals of $[\alpha, \beta]$ with $0 < \varepsilon$. Define

$$v_c(I, J) := \frac{\int_I \rho_c(t)\, dt}{\int_J \rho_c(t)\, dt}.$$

To be able to use Lemma 22, we suppose that $\varepsilon < \min(\frac{1}{4}, \exp\frac{-4}{(1-\beta)^2})$. Now $\delta := \sqrt{\varepsilon} - 2\varepsilon > 0$.

Let us fix an arbitrary $c \in [\alpha, \beta]$ and let $d := \max(c-\delta, \alpha)$ and $e := \min(c+\delta, \beta)$.

Case 1: Suppose that $(I \cup J) \subset [\alpha, d]$.

The function

$$h(t) := \int_c^1 \frac{1}{s-t} f(s)\, ds$$

is increasing on $(0, c)$, therefore

$$h(u-\varepsilon)\int_I a(t)\, dt \leqslant \int_I \phi_c(t)\, dt \leqslant h(u)\int_I a(t)\, dt$$

and

$$h(u)\int_J a(t)\, dt \leqslant \int_J \phi_c(t)\, dt \leqslant h(u+\varepsilon)\int_J a(t)\, dt$$

from which

$$v_c(I, J) \leqslant \frac{\int_I a(t)\, dt}{\int_J a(t)\, dt} \quad \text{and} \quad v_c(J, I) \leqslant \frac{h(u+\varepsilon)\int_J a(t)\, dt}{h(u-\varepsilon)\int_I a(t)\, dt}.$$

But $a(t)$ has smooth integral on $[\alpha, \beta]$. Also

$$h(u+\varepsilon) \leqslant \left(1 + \frac{2\varepsilon}{c-(u+\varepsilon)}\right) h(u-\varepsilon),$$

which can be gained by integrating the following inequality with respect to s from c to 1:

$$\frac{1}{s-(u+\varepsilon)}f(s) \leqslant \left(1+\frac{2\varepsilon}{c-(u+\varepsilon)}\right)\frac{1}{s-(u-\varepsilon)}f(s).$$

But $c-(u+\varepsilon) \geqslant \delta$, so $h(u+\varepsilon)/h(u-\varepsilon) \leqslant 1+2\varepsilon/(\sqrt{\varepsilon}-2\varepsilon) = 1+o(1)$.

Thus we have proved that $v_c(I,J) = 1+o(1)$ and $v_c(J,I) = 1+o(1)$ as $\varepsilon \to 0$, where $(I \cup J) \subset [\alpha, d]$.

Case 2: Suppose that $(I \cup J) \subset [e, \beta]$. Exactly as in case 1, we can see that $v_c(I,J) = 1+o(1)$ and $v_c(J,I) = 1+o(1)$ as $\varepsilon \to 0$, where $(I \cup J) \subset [e, \beta]$.

Case 3: Now suppose that $(I \cup J) \cap (d,e) \neq \emptyset$. Let $d' := \max(c-\sqrt{\varepsilon}, \alpha)$ and $e' := \min(c+\sqrt{\varepsilon}, \beta)$. Because of our assumption, $I \cup J \subset [d', e']$.

Let first $t \in (d', c)$ be arbitrary and define the function $h(t)$ as in case 1.

From Lemma 22, we have

$$\left|\frac{\phi_c(t)}{-a(t)\ln(c-t)} - f(c)\right| \leqslant \frac{(1+\frac{1-c}{2})(L+f(c)) + ||f||_{L^1[c,1]}}{\sqrt{-\ln(c-t)}} \tag{21}$$

where L is the Lipschitz constant of f on the interval $[c, \frac{c+1}{2}]$. But $c \in [\alpha, \beta]$, so

$$L \leqslant \sup_{\substack{x,y \in \left[\alpha, \frac{\beta+1}{2}\right] \\ x \neq y}} \left|\frac{f(y)-f(x)}{y-x}\right|$$

is a finite global upper bound for the possible Lipschitz constants. Also, $f(c) \leqslant ||f||_{[\alpha,\beta]}$ and $||f||_{L^1[c,1]} \leqslant ||f||_{L^1[-1,1]}$. Therefore, the numerator of (21) is bounded by a global constant (which depends on the function $f(s)$ only).

Since now $t \in (d', c)$, $c-t \leqslant \sqrt{\varepsilon}$, so $1/\sqrt{-\ln(c-t)} = 1+o(1)$ as $\varepsilon \to 0$. From these, we can conclude that

$$\frac{\phi_c(t)}{-a(t)\ln(c-t)} - f(c) = o(1) \quad t \in (d', c) \text{ as } \varepsilon \to 0. \tag{22}$$

We also know that $a(t)$ is a continuous function, so

$$a(t) = a(c) + o(1) \quad \text{for } t \in (d', c) \text{ as } \varepsilon \to 0. \tag{23}$$

$a(t)$ and $f(t)$ are bounded on $[\alpha, \beta]$ and so from (22) and (23) we can see that

$$\phi_c(t) = (a(c)f(c) + o(1)) \ln\frac{1}{c-t}$$

for $t \in (d', c)$ as $\varepsilon \to 0$.

Now if $t \in (c, e')$, the same argument shows that

$$\psi_c(t) = (a(c)f(c) + o(1)) \ln\frac{1}{t-c}$$

for $t \in (c, e')$ as $\varepsilon \to 0$. (Here we used the assumption, that $b(c)g(c) = a(c)f(c)$.)

Putting these together, we have

$$\rho_c(t) = (a(c)f(c) + o(1)) \ln \frac{1}{|c-t|}$$

for $t \in (d', e')$ as $\varepsilon \to 0$.

Since the functions $a(c)$ and $f(c)$ have a positive lower bound on $[\alpha, \beta]$, we get

$$v_c(J, I) = \frac{a(c)f(c) + o(1) \int_I \ln \frac{1}{|c-t|} dt}{a(c)f(c) + o(1) \int_J \ln \frac{1}{|c-t|} dt} = (1 + o(1)) \frac{\int_J \ln \frac{1}{|c-t|} dt}{\int_I \ln \frac{1}{|c-t|} dt}.$$

By Proposition 20, $\log(1/|x|)$ has smooth integral on any interval $[-a, a]$, $0 < a < 1$. Thus the second factor is $1 + o(1)$ as $\varepsilon \to 0$, so

$$v_c(J, I) = 1 + o(1) \quad \text{where } I, J \subset (d', e') \text{ as } \varepsilon \to 0,$$

and similarly

$$v_c(I, J) = 1 + o(1) \quad \text{where } I, J \subset (d', e') \text{ as } \varepsilon \to 0.$$

Cases 1–3 together proves that the family of functions $\mathscr{F}$ has uniformly smooth integral on $[\alpha, \beta]$. $\square$

Lemma 24. *Let H be a monotone increasing function on $(-1, 1)$ which is absolutely continuous inside $(-1, 1)$ and for which $H(s)/\sqrt{1-s^2} \in L^1[-1, 1]$. Define*

$$v_c(x) := -PV \int_{-1}^{c} \frac{1}{(s-x)\sqrt{1-s^2}} \, ds \quad c \in [-1, 1], \; x \in (-1, 1)$$

(*which is a principal value integral, if $x < c$*). *Then we can integrate by parts as follows*:

$$PV \int_{-1}^{1} \frac{H(s)}{(s-x)\sqrt{1-s^2}} ds = \int_{-1}^{1} v_s(x) \, dH(s), \quad a.e. \; x \in [-1, 1]. \tag{24}$$

Proof. In fact (24) is true whenever $H'(x)$ exists at x. Since the derivative of H exists almost everywhere, this will prove the lemma.

Because of (17) we can define $v_c(x)$ with regular integrals, too:

$$v_c(x) = \begin{cases} -\int_{-1}^{c} \dfrac{1}{(s-x)\sqrt{1-s^2}\, ds} & \text{if } c < x, \\ \int_{c}^{1} \dfrac{1}{(s-x)\sqrt{1-s^2}\, ds} & \text{if } x < c \end{cases}$$

$$PV\int_{-1}^{1}\frac{H(s)}{(s-x)\sqrt{1-s^2}}\,ds$$
$$=\lim_{\varepsilon\to 0^+}\left[\int_{-1}^{x-\varepsilon}\frac{H(s)}{(s-x)\sqrt{1-s^2}}\,ds+\int_{x+\varepsilon}^{1}\frac{H(s)}{(s-x)\sqrt{1-s^2}}\,ds\right].$$

Here $\int_{-1}^{x-\varepsilon}=\int_{a}^{x-\varepsilon}+\int_{-1}^{a}$ and $\int_{x+\varepsilon}^{1}=\int_{x+\varepsilon}^{b}+\int_{b}^{1}$ where the second terms tend to 0 as $a\to-1^+$, $b\to 1^-$. Integrating by parts, we get

$$\int_{a}^{x-\varepsilon}\frac{H(s)}{(s-x)\sqrt{1-s^2}}\,ds+\int_{x+\varepsilon}^{b}\frac{H(s)}{(s-x)\sqrt{1-s^2}}\,ds$$
$$=-v_{x-\varepsilon}(x)H(x-\varepsilon)+v_a(x)H(a)+\int_{a}^{x-\varepsilon}v_s(x)\,dH(s)$$
$$-\ v_b(x)H(b)+v_{x+\varepsilon}(x)H(x+\varepsilon)+\int_{x+\varepsilon}^{b}v_s(x)\,dH(s).$$

We will show that $\lim_{a\to-1^+}v_a(x)H(a)=0$, $\lim_{b\to 1^-}v_b(x)H(b)=0$, and if $H'(x)$ exists at x, then $\lim_{\varepsilon\to 0^+}[v_{x+\varepsilon}(x)H(x+\varepsilon)-v_{x-\varepsilon}(x)H(x-\varepsilon)]=0$.

From these the statement of the lemma follows:

We have

$$v_{x+\varepsilon}(x)H(x+\varepsilon)-v_{x-\varepsilon}(x)H(x-\varepsilon)$$
$$=[v_{x+\varepsilon}(x)-v_{x-\varepsilon}(x)]H(x+\varepsilon)-v_{x-\varepsilon}(x)[H(x-\varepsilon)-H(x+\varepsilon)].$$

Here $\lim_{\varepsilon\to 0^+}[v_{x+\varepsilon}(x)-v_{x-\varepsilon}(x)]=PV\int_{-1}^{1}\frac{1}{(s-x)\sqrt{1-s^2}}\,ds=0$, $H(x+\varepsilon)$ is bounded as $\varepsilon\to 0^+$ and

$$\lim_{\varepsilon\to 0^+}\frac{H(x-\varepsilon)-H(x+\varepsilon)}{2\varepsilon}\to -H'(x).$$

We can also see, that $\lim_{\varepsilon\to 0^+}\varepsilon v_{x-\varepsilon}(x)=0$, since

$$0\leqslant v_{x-\varepsilon}(x)=\int_{-1}^{x-\varepsilon}\frac{1}{(x-s)\sqrt{1-s^2}}\,ds$$
$$\leqslant C_1\int_{-1}^{x-\varepsilon}\frac{1}{x-s}\,ds\leqslant C_1\ln\frac{1}{\varepsilon}+C_2.$$

It has remained to prove that $\lim_{a\to-1^+}v_a(x)H(a)=0$ (the proof of $\lim_{b\to 1^-}v_b(x)H(b)=0$ is the same). First notice that $v_a(x)$ behaves like $C\sqrt{(1+a)}$ as $a\to-1^+$. Therefore we are done, if we can prove the following: If h is a monotone function for which $h(s)/\sqrt{s}\in L^1[0,1]$, then $\lim_{s\to 0^+}\sqrt{s}h(s)=0$. Indeed, if this limit is not zero, then there exist a $0<\delta$ and a decreasing sequence $s_n\to 0$: $\delta/\sqrt{s_n}\leqslant|h(s_n)|$. We can suppose that $|h(s)|$ is decreasing in $(0,\rho)$ for some ρ. (The increasing case is obvious.) We can also suppose that $s_1 < \rho$. Now

$$\int_0^\delta \frac{|h(s)|}{\sqrt{s}}\, ds \geqslant \sum_1^\infty (s_n - s_{n+1}) \frac{\delta}{s_n} = \delta \sum_1^\infty \left(1 - \frac{s_{n+1}}{s_n}\right).$$

But this infinite sum is infinity, since $\prod s_{n+1}/s_n = 0$. This contradicts $h(s)/\sqrt{s} \in L^1[0,1]$. $\square$

At last we are in the position to finish the proof of Theorem 6. We already have proven the first part of the theorem (see Lemma 17). It remains to show that the density of the equilibrium measure has smooth integral inside (a,b). We have seen that it is enough to give a proof when the support is an interval, that is:

If $S_w = [a,b]$ and Q is weak convex on (a,b), then the density v has smooth integral inside (a,b).

(Although by the balayage process we achieved that $\Sigma = [a,b]$, in this statement Σ does not have to be $[a,b]$. All we need is S_w to be an interval. We also remark that although the proof will be short, everything we proved so far is used in the proof.)

Proof. We can suppose that $a = -1$, $b = 1$. Let $[\alpha, \beta] \subset (-1,1)$ be an arbitrary interval.

Let $\kappa \in [-1,1]$ be a number for which

$$Q'(x) \leqslant 0 \quad x \in (-1, \kappa),$$

$$Q'(x) \geqslant 0 \quad x \in (\kappa, 1).$$

Let us define

$$F(s,t) := \begin{cases} \dfrac{1+t}{\pi^2\sqrt{1-t^2}}(1-s)Q'(s) & \text{if } s \in (-1,\kappa), \\ \dfrac{1-t}{\pi^2\sqrt{1-t^2}}(1+s)Q'(s) & \text{if } s \in (\kappa, 1). \end{cases}$$

If in (18) we choose $r := \kappa$, we get

$$v(x_0) = PV \int_{-1}^1 \frac{F(s,x_0)}{(s-x_0)\sqrt{1-s^2}}\, ds - \frac{E}{\sqrt{(1-x_0^2)}}, \quad x_0 \in (-1,1),$$

where $E \in \mathbb{R}$ is some constant, so by Lemma 24:

$$v(x_0) = \int_{-1}^1 v_s(x_0)\, dF(s,x_0) - \frac{E}{\sqrt{(1-x_0^2)}},$$

where the variable of the integration is s, and

$$v_s(x_0) := -PV\int_{-1}^{s}\frac{1}{(t-x_0)\sqrt{1-t^2}}\,dt.$$

Therefore

$$\begin{aligned} v(t) = & \frac{1+t}{\pi^2\sqrt{1-t^2}}\int_{-1}^{\kappa} v_s(t)d[(1-s)Q'(s)] \\ & + \frac{1-t}{\pi^2\sqrt{1-t^2}}\int_{\kappa}^{1} v_s(t)d[(1+s)Q'(s)] - \frac{E}{\sqrt{1-t^2}} \quad t\in(-1,1). \end{aligned} \tag{25}$$

Setting $f(s) := g(s) := 1/\sqrt{1-s^2}$ and $a(t) := b(t) := \frac{1+t}{\pi^2\sqrt{1-t^2}}$ in Lemma 23, we gain that the family of functions

$$\left\{ t \mapsto \frac{1+t}{\pi^2\sqrt{1-t^2}} v_c(t):\ c\in[-1,1] \right\}$$

has uniformly smooth integral on $[\alpha,\beta]$. So by Proposition 18, the first term of (25) has smooth integral on $[\alpha,\beta]$. Similarly the second term of (25) has also smooth integral on $[\alpha,\beta]$.

Since $v(t)$ has a positive lower bound on $[\alpha,\beta]$, no matter what the sign of E is, $v(t)$ has a smooth integral on $[\alpha,\beta]$ by Propositions 18 and 19. This completes the proof of Theorem 6. $\square$

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